The Search for the Diodorean Frame

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ABSTRACT

Diodorean modalities are logical notions that specify, in a precise way, how sentences may be true with respect to time: a sentence is diodoreanly necessary at a given instant iff it is true since that instant on. Arthur Prior has treated them as sentential operators and built up a logic for such modalities (**DIOD**) conjecturing that the *frame for* such a logic (the "diodorean frame") was the frame for **S4**. The Conjecture was soon proved false, through a number of counterexamples that played a role in the research on modal logics between **S4** and **S5**. The present paper aims at showing that (i) the search for the diodorean frame benefited from such a research, and that (ii) there has been a mutual interaction between the search of the diodorean frame and some characterisation results. The paper is divided into five parts. In section 1, I will introduce diodorean modalities, while in Section 2 I will be focusing on Prior's reconstruction of the Master Argument and his characterisation of **DIOD**. In section 3, I present a conjecture Prior advanced about the characterisation of **DIOD** and some counterexamples to it. The notions of "*frame*" and "*frame for*" will be also introduced. In section 4 I summarise the connections between the search of the diodorean frame and some researches in modal logic. Section 5 presents a short conclusion.

1. INTRODUCTION

Diodorean possibility and *necessity* constitute the *diodorean modalities*, and are defined as follows: *p* is *diodoreanly possible* (since now on, *d-possible*) at a given instant *t* iff *p* is true at *t or* at *some* later instant, and *p* is *diodoreanly necessary* (since now on, *d-necessary*) at a given instant *t* iff *p* is true at *t* and at *every* later instant.

d-necessity and d-possibility are comprised in the family of modalities, *i.e.* those notions that specify the truth-value of sentences in a non-extensional way. A list of such notions usually include epistemic and doxastic predicates ("... is believed", "... is known"), notions as "possible" and "necessary". Remarkably, also *tenses* are included in the list, since they specify the way a sentence is true with respect to time.

Today all these notions receive an essentially uniform treatment as *sentential operators*, that is operators that transform sentences in other sentences (e.g. "I eat" in "It is possible that I eat"). Such an approach is due to the work of Saul Kripke[1], and is considered one of the major results of contemporary logic. The *operators* that aim at expressing *tenses* are called *temporal*

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[1] (Kripke, 1959) is a milestone of modal logic. There, Kripke focuses on the alethic notion of necessity, but in doing this he provides a semantics that has soon been used for any kind of modality (temporal, epistemic, doxastic, deontic). The formal tools used to Kripke are now the standard ones in modal logic, and I will employ them in the present paper.

operators, and the logics and languages that comprise them are called *temporal logics*[2] and *temporal languages*. When referring to the field in its entirety, the label "*temporal logic*" is often used.

In what follows, we will deal with temporal logic, since diodorean modalities have been defined on the basis of temporal notions. The temporal language will be given in section 2.1.

Throughout the paper we will also meet logics where the notions of logical or metaphysical "necessity" and "possibility"[3] are expressed. They are the so-called "*alethic (modal) logics*". The main difference between these logics and the temporal ones is that, while

$\bullet p \rightarrow p$

is taken as valid when "•" is given an "alethic" reading (as "necessarily"), it is invalid in any plausible temporal reading[4]. Today, "alethic logics" are simply called "*modal logics*". This may create some ambiguities, because temporal logics are often referred to as "modal logics".

In this paper, I will mainly use the label "modal logics" for those logics whose operators can be read in an alethic way, and "temporal logics" for those that fit with a temporal reading. In section 3.1 my use of "modal" will be more ambiguous, but what I say there applies to both temporal and modal logics.

A neat difference in label may not mirror a dramatic divide in the labelled subjects. This is exactly the case with temporal and modal logics: some alethic logics may be built up as fragments of temporal logics. In other words, we may take a temporal logic **L** and define there a "modal operator" on the basis of temporal operators. Then we may extract a modal logic **L'** by taking that fragment of **L** that contains all and only the sentences where just the modal operator appear[5]. Thus, there may be no real divide: some alethic logics may have a temporal reading.

It is easy to see that a logic for diodorean modalities will have this reading, since d-necessity and d-possibility are defined by temporal notions. However, I consider the diodorean logic as a fragment where only operators for d-necessity and d-possibility. In the paper I will also mention two modal logics: **S4** and **S5**. As we will see, **S4** has been related with diodorean

[2] Some insisted on the opportunity of calling them *tense* operators and *tense* logics, in order to distinguish them from other logics that express relations between instants, or between sentences and instants) and drop away tenses. These logic would be "temporal logics" (since dealing with time), but not *tense logics*. However, today the label "*temporal logics*" is used for both of them. It is clear by what follows, that here take into account only logics containing sentential operators for tenses.

[3] The notion of a "logical necessity (possibility)" can be characterised as follows: *p* is a logically necessary (possible) truth if it follows from (is compatible with) the laws of logic. The notion of a "metaphysical necessity" strikes many as unclear. Probably, one of the most perspicuous characterisation is: *p* is a metaphysically necessary (possible) truth iff *p* is true in virtue of the objects it is about (if it is compatible with the nature of the object it is about).

[4] "If always (at least once) in the future (past) [it is the case that] *p*, then *p*" is clearly false.

[5] To be more precise, we take the fragment that contains all and only the sentences where there are just those combinations of tenses that define the modal operator.

modalities, and both logics are very important in other fields of logics: they have been used to study the relations between Intuitionistic logic, Classical logic and logics between the two.

As many modal notions, diodorean modalities have philosophical roots. They represent a view on possibility that was supported in Antiquity, and they was involved in the debates about determinism and free will. Arthur Prior rediscovered such modalities in the half of the past century, and propose a formal approach to them[6].

In what follows, my aim is to highlight the (usually) neglected connections between some researches in modal logic -in the Fifties and the Sixties- and the search for the *diodorean frame*. More precisely, I will show the benefits that Prior's investigation received by researches in *characterisation results* (see Section 3) and in the logics between **S4** and **S5** (that since now on I will call "*intermediate logics*"). Before doing this, in Section 2 I present Prior's reconstruction of the Master Argument, an argument that the greek philosopher and logician Diodorus Cronus used to support the view that gave rise to the modalities that bring his name. Some assumptions made by Prior decisively influenced the search for the diodorean frame. Section 3 presents a conjecture Prior advanced about the characterisation of the diodorean logic **DIOD**, and introduces as well counterexamples to this conjecture. This counterexamples shed light on the existence of previously unknown logics. Section 4 summarises the content of the paper, and is followed by a short conclusion (Section 5). A last remark before starting. In all his works, Prior uses the so-called *polish notation*, a symbolism where logical connectives are prefixed to the sentences they connect. Such a notation is very hard to read and somewhat unfamiliar today. For these reasons, in this paper I will use the contemporary notation (with connectives appearing *in* the sentences, and not prefixed to them).

2. FROM THE MASTER ARGUMENT TO THE DIODOREAN LOGIC

Diodorean modalities have been named thus after the ancient greek philosopher and logician Diodorus Cronus, who defended his conception of modalities in an argument that became famous as "the Master Argument" (see (Denyer, 2009), this volume). By the latter, Diodorus aimed at showing that the only plausible meaning of "possible" is "true either now or at least once in the future". The Master Argument is well-known to us for its quite puzzling character: we have just indirect sources of it, and all of them mention two premises and the supposed conclusion, without reporting the inference from the former to the latter. This is quite problematic, since any reasonable derivation of the conclusion from the given premises seems to require further assumptions.

The premises are:

a) If a sentence *p held*, then it is a matter of necessity that it *held* (the past is somehow necessary).

[6] In (Prior, 1955).

b) If necessarily $\neg q$ and $p \rightarrow q$, then necessarily $\neg p$.

and the conclusion is:

z) if *p* is and will always be false, then *p* is impossible.

In other words, by assuming ***a)*** and ***b)***, it should derive that the notion of "necessity" appearing in the premises collapses on the notion of "d-necessity". Here, it is worth noting that the notion of necessity employed in the premises and steps of the Argument is not d-necessity. Indeed, the aim of the argument is reducing an otherwise characterised notion of modality to the diodorean one, and employing the latter as "the" notion of modality through the Argument would jeopardise it with circularity. This fits well with the fact that the Argument supports the Diodorean Modalities as the right way of conceiving "modalities" (intended on a more general way), and it does not aim at proposing the *definition* of the d-modalities.

The success of the Master Argument is ascribable to the fact that ***a)*** and ***b)*** were widely accepted by ancient philosophers, and that the lost inference from them to the conclusion was considered correct. To us, the problematic aspect of the Argument is due to the fact that we have no direct or decisive evidence for guessing how it should be suitably restored. The combination of the two things has promoted a variety of attempts to reconstruct the Argument in order to make its inferences explicit. Even if we restrict ourselves to the past century, a wide number of such attempts have been proposed, and the debate about them is lively. In addition, since a decisive evidence about the Argument lacks, it is difficult to foresee a conclusion.

Reconstructions of the Master Argument have today the form of precise formalisations[7]. This situation is due to the fact that for about ten years (1955-1965) the Master Argument has held a main position in the crossover field of philosophy and modal logics. This field was fed by traditional philosophical problems but was concerned at the same time with the properties of the formal logics and languages that were employed at that time to shed light on the notions of necessity, eternity, knowledge and the like. Diodorean modalities and the Master Argument have been a very specific example of how philosophical problems have been readdressed by formal tools.

Since we are here interested in diodorean modalities and their connections with the search on frames for modal logics, I will focus just on Prior's reconstruction[8]. Indeed, it has been the starting point of a work on a diodorean logic and on the frame for it (see Section 3.1 and 3.2).

[7] For this aspect, see (Denyer, 2009), this volume.
[8] For comparing Prior's reconstruction with other formal attempts, see (Denyer, 2009), this volume.

2.1 PRIOR'S RECONSTRUCTION OF THE MASTER ARGUMENT

Arthur Prior is the first to put together the exegetical problem and the tools of contemporary modal logic. First he deals with diodorean modalities in (Prior, 1955), where he provides a characterisation of these modalities and his reconstruction of the Master Argument by giving the guidelines he will always follow thereafter. Then, he comes back on the issue in various papers or book's chapters (see (Prior, 1958), (Prior, 1962), (Prior, 1957) and (Prior, 1967)), often correcting or making more precise what he had previously stated.

In order to give a perspicuous and straightforward description of Prior's reconstruction, we need to introduce a formal tool: a language that is able to express tenses and give the definition of the diodorean modalities. The temporal language (l_T) we need is an expansion of the language of propositional classical logic by the operators *P* and *F*. *P* and *F* mean "at least once in the past" and "at least once in the future", respectively. *H* and *G* are their *duals*[9], to be read as "always in the past" and "always in the future". Thus, *PGp* means "at least once in the past [it is the case that] always in the future *p*"[10]. Let me use *L* for the operator of d-necessity, that is defined as follows:

$Lp := p \wedge Gp$

L will be the operator of d-necessity (do be defined as *M*'s dual)[11]. (l_T) must include as well a way for expressing the notion of necessity involved in the premises of the Argument. As we have seen, it cannot be expressed by *L*, on the pain of circularity. Ancient sources give us no precise hint on how to interpret such a notion[12], but it is clear that it should have an intuitive or theory-laden reading (*i.e.* it should correspond to some common or philosophical view on necessity). Indeed, the Argument is interesting as far as it reduces to d-necessity an *otherwise conceived* notion of necessity, as I have already suggested above. If this was denied, the Master Argument would loose any intuitive or philosophical appeal.
Since facing the Master Argument and its problems is beyond the tasks of this work, I will use here the symbol **NEC** for expressing the necessity to be reduced, while keeping myself neutral on the viable interpretations of it. In conformity with the contemporary modal machinery, **NEC** will be treated as an operator. In addition it is implicit in Prior's reconstruction (as in any other one), that the symbol obeys the rules of inference:

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[9] In symbols: $Hp := \neg P \neg p$ and $Gp := \neg F \neg p$

[10] Another example: *HFp* is "always in the past [it is the case that] at least once in the future *p*" is true. Notice that this sentence is nothing but $\neg P \neg Fp$, and its negation is consequently $PG\neg p$.

[11] In symbols: $Mp := \neg L \neg p$.

[12] In any case, the reduction looks implausible if logical necessity involved in the premises: ***a)*** would not sound feasible, and in any case the conclusion would sound hardly acceptable if it was "*p* is true by virtue of the laws of logic iff *p* is and will always be true". In addition, it is clear that the notion to be reduced is not the notion of "possibly" as "at least once", because if "necessary" is read as "always", then the premise ***a)*** is patently false: we may have that *p* was true and that, in some earlier instant, *p* *had been* false up to that instant.

RNEC $\vdash p \;\Rightarrow\; \vdash$ **NEC**p

(in other words, if p is a theorem, also **NEC**p is) and:

MP $\vdash (p \rightarrow q) \wedge p \Rightarrow \vdash q$

US $\vdash p \Rightarrow \vdash q$ form substitution of propositional variables $r_1, ..., r_n$ in p with formulae whichever $b_1, ..., b_n$

(Modus Ponens and Uniform Substitution). In symbols, ***a)*** and ***b)*** become:

a') $Pq \rightarrow$ **NEC**Pq

b') **NEC**$(p \rightarrow q) \wedge$ **NEC**$\neg q) \rightarrow$ **NEC**$\neg p$

a')-b') contribute to settle the general framework on which the Argument has to run, and yet tell us nothing of the "inferential gap" that stands between the Argument's premises and its conclusion. To restore the Argument, Prior added two premises to those mentioned by the ancient sources:

c) $PG\neg p \rightarrow \neg p$

d) $(\neg p \wedge G\neg p) \rightarrow PG\neg p$

It is worthy to include premise ***c)*** in the set of sentences that should hold under the Diodorean conception of truth in time. Indeed, it is part of our most basic intuitions about time that, if once in the past [it is the case that] p is going to be true at any subsequent instant, then p is true now (otherwise $\neg Gp$ should hold at any instant previous than now). Things are not that easy for ***d)***, as we shall see below. Once this is settled, Prior's reconstruction reshapes the Argument as follows:

1	$(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)$	by propositional logic
2	$(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$	by propositional logic
3	$PG\neg p \rightarrow$ **NEC**$PG\neg p$	by ***a')***, with $G\neg\, p$ substituting q
4	$(\neg p \wedge G\neg p) \rightarrow$ **NEC**$PG\neg p$	by **1**, **3** and ***d)*** via **MP**, with $\neg p \wedge G\neg p$ substituting P, $PG\neg p$ substituting q and **NEC**$PG\neg p$ substituting r
5	$(\neg p \wedge G\neg p) \rightarrow$ (**NEC**$(PG\neg p \rightarrow \neg p)$)	by ***c)*** and **1** via **RNEC** and **MP**
6	**NEC**$(PG\neg p \rightarrow \neg p)$	by ***c)*** via **RNEC**
7	$(\neg p \wedge Gp) \rightarrow$ **NEC**$\neg p$	by **3**, **5**, **6**, ***b')*** via **MP** and **RNEC**
8	$L\neg p \rightarrow$ **NEC**$\neg p$	by **7** and the definition of L

Thus, if ***a')-d)***, **RNEC** and **MP** are embarked together, then Diodorus' reduction follows. No doubt can be cast on the validity of the argument.

Nevertheless, some perplexities may arise if we consider premise ***d)***. Indeed, ***d)*** is valid only if time is *discrete*. Suppose that time is dense or continuous, and that p is false from t on. Now take any instant t' earlier than t. Since there are infinite instants between t' and t, we cannot exclude that p is true in one of such instants, say t''. The same for t'' and t, and so on. Thus, $PG\neg p$ and hence ***d)*** are falsified. On the contrary, if time is discrete t must have an immediate predecessor. The above situation standing, the predecessor of t verifies $PG\neg p$, since p is false from t on. Thus ***d)*** is verified. Useless to say, the imposition of a discrete time cannot but rise doubts. However, it seems plausible in a reconstruction of the Argument. Indeed, there is some evidence that Diodorus proposed a form of temporal atomism that included the discreteness of time[13].

It is not the aim of this paper to determine how this should influence our evaluation of Prior's attempt[14]. The main point here is that *discreteness* had a major historical role in dismissing a conjecture that Prior advanced about the frame for a diodorean logic, and that I will introduce in the next section. Consequently, the acceptance of it had an influence in the search of the diodorean frame. In other words: the inclusion of discreteness in Prior's reconstruction of the Argument has been a reason for conceiving diodorean modalities as satisfying them, and consequently for looking at a frame where the condition is fulfilled.

2.2 THE DIODOREAN LOGIC

On the basis of his reconstruction, Prior outlined a logic for the diodorean modalities, *i.e.* a logic where all and only the diodorean tenets (as emerging by Prior's reconstruction) and their consequences where theorems. This is the main task of (Prior, 1955) and (Prior, 1958), and one of the main topics in (Prior, 1957) and (Prior, 1967). The logic was meant to be a modal logic based on a temporal one, and this is one of the reasons for some confusion we find in the above texts. Indeed, Prior insists on the temporal character of diodorean modalities, but at the same time the frame he proposes for them (see section 3.1) is not suitable for temporal logics (for the notion of frame, see again section 3.1). Thus the reader may have the impression that Prior stresses the "temporal meaning" of diodorean modalities just when he deals with them in a non-formal way. When formal topics are considered, Prior seems to treat them with no regard to such a "meaning". This is due to the fact that, when explaining what diodorean modalities are, he presents them through the notions of presentity and futurity. Otherwise, it would be difficult to understand the rationale of introducing them among the modal notions. On the contrary, Prior considered diodorean modalities "in isolation" (as they were joined by no tense operator or defined by no temporal notion) when he aimed at investigating their

[13] See (Denyer, 1981).
[14] In any case, Prior's reconstruction is still one of the most convincing. For this, see (Denyer, 2009), this volume.

formal properties. This is clear by the fact that, when Prior writes about the diodorean logic, he describes it as a logic where just *L* and the dual *M* are the operators.

This twofold approach to them should not induce us to believe that a real division holds here. Indeed, for Prior the diodorean logic should be in accordance with the properties of time that make the Master Argument valid. This is clear by the fact such an accordance is used by Prior to admit or dismiss hypothesis on the diodorean frame. In proposing the principles of such a logic, Prior relied on a very basic intuition about time: the earlier/later relation between instant is *transitive*. Obviously, *discreteness* must be imposed, for the reasons I have suggested in the above section. Given this, Prior settled the following principles settled for the *tense operators* and the *diodorean modalities*:

AG1	$G((p \rightarrow q) \wedge Gp) \rightarrow Gq$	and mirror image
AG2	$Gp \rightarrow GGp$	and mirror image
AG3	$(p \wedge Gp) \rightarrow PGp$	and mirror image
AG4	$PGp \rightarrow p$	and mirror image
AG5	$Gp \rightarrow Fp$	and mirror image
AL1	$L((p \rightarrow q) \wedge Lp) \rightarrow Lq$	
AL2	$Lp \rightarrow p$	
AL3	$Lp \rightarrow LLp$	

together with the following rules of inference:

MP as above.
RG $\vdash p \Rightarrow \vdash Gp$
RL $\vdash p \Rightarrow \vdash Lp$

where the mirror image of a sentence *p* is the result of substituting any occurrence of *P* (or *F*) with *F* (or *P*). **AG1** together with **RG** corresponds to the condition that is usually called *normality*[15], and its presence in **DIOD** is justified by the fact that it was allegedly accepted by the greek logicians. **AG2** is due to the transitivity of the earlier/later relation on time, while **AG3** expresses in the language the discreteness of time. **AG4** is premise ***c)*** under the substitution of $\neg p$ by *p*. **AG5** expresses the infinity of time: if *every* instant later than *t* verifies *p*, *then there is* an instant later than *t* that verifies *p*. If time had an end, this would not be true: in this case, *t* could be the last instant, *Gp* would be vacuously true at it. Indeed, no instant later than *t* would falsify *p*, since there is no such instant. But for the same reason, *Fp* would not be true. The infinity of time may be found questionable. However, Prior explicitly embarked it[16], and I will follow him on this point. In dealing with the Argument, Prior does

[15] Normal temporal logics are those logics where $(\bullet(p \rightarrow q) \wedge \bullet p) \rightarrow \bullet q$ is valid (where $\bullet$ is *G* or *H*).
[16] The matrix Prior uses in (Prior, 1957) to represent diodorean modalities is infinite, and since each number of the matrix should be read as if it is associated to an instant, we must conclude that the matrix suggests a reading of time where infinity is comprised.

never hint for some form of "non-homogeneity" between the past an the future, this meaning that validity might not be preserved by the mirror image of a sentence. Thus, I the mirror images of **AG1**-**AG5** to be valid. The need of including **AL1** among the principles is clear by the Master Argument: since (**NEC**$(p \rightarrow q) \wedge$ **NEC**$\neg q) \rightarrow$ **NEC**$\neg p$ and **NEC**$p \leftrightarrow Lp$ are valid in the diodorean perspective, one can easily infer that $(L(p \rightarrow q) \wedge L\neg q) \rightarrow L\neg p$ is valid too. But it is easy to see that the latter is equivalent to **AL1**. This proves as well the validity of **AG1**[17]. **AL2** is made valid by the definition of *L*, since *Lq* is nothing but $p \wedge Gp$ and $(p \wedge Gp) \rightarrow p$ is valid. **AL3**'s validity is due to the definition of *L* and **AG2**. The validity of the rules may be maintained on the ground of what we know about logic in (greek) antiquity (Modus Ponens was universally taken as a correct rule, a sentence that is proved to be true was taken *ipso facto* as always true).

The axiomatic and inferential apparatus settled above is enough to build a diodorean logic. However, before doing this, something must be said on how Prior read **AG4**. Suppose that each instant may be followed by different, incompatible courses of events. Each "course of events" (or *branch*) is made by linearly ordered instant and is maximal w.r.t. such instants[18]. Well, how should we read $PGp \rightarrow p$ in this case? If we conceive time as linear, reading the sentence is straightforward, but if time branches in the future, the sentence may look ambiguous. What does its antecedent mean? It means that there is a past such that *Gp* is true with respect to a given instant *and a given branch (or all branches)*? Or does it mean that there is a past such that *Gp* is true with respect to a given instant *and some branch*[19]? According to Prior, **AG4** should be read on a linear time. However, the linearity of time is usually taken as the main way of representing determinism, that is (in temporal contexts), the view that

DET There is no alternative to what happens, happened or will happen.

In other words, not only the past and the present are beyond any possible attempt to modify them: also what will happen is completely determined[20]. The link with linear time is

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[17] Indeed, by **AL1** and the definition of *L*, it derives that $(((p \rightarrow q) \wedge p) \rightarrow q) \wedge (G((p \rightarrow q) \wedge Gp) \rightarrow Gq)$, by which **AG1** follows.

[18] This means that if *b* is a branch, then for every *t* and *t'*, if they belong to *b*, they are *comparable* (*i.e.* the one is either earlier, or later than the latter, or they are the same instant).

[19] In the first case, **AG4** is true, while in the second it is false: if things *could have gone* as verifying *p* forever after a certain instant, this does not mean that they *have gone* in such a way. Hence we could have $PGp \wedge \neg p$. Today we have a number of different semantics that allow us to express all this options. *Ockhamist semantics* are able to express all options: at a given instant *t* and w.r.t. the branch *b*, "*Gp*" is read "in every instant later than *t* and belonging to *b*, *p* is true", while "in every instant later than *t* and belonging to all (some) *b*, *p* is true" is expressed by $\neg\Diamond\neg Gp$ ($\Diamond Gp$), respectively. For these semantics and their developments, see (Zanardo, 2009) and (Øhrstrøm, 2009), this volume. The first, important work on semantics for non-linear time has been carried out by Prior. A good overview of this work is present in (Prior, 1967).

[20] When embarking time-reduced modalities as we are doing here, determinism should not be confused with the idea that $Mp \rightarrow Lp$. The latter is stronger than determinism, since stating that what happens now or later, always happens in the future (or that what sometimes happens, always happens, if "possible" is read as "at least once in time".

straightforward: given t and t' , either they are identical, or the one is earlier or later than the latter, to the effect that any instant is followed *only by one* "possible development" of the events[21].

Determinism and linearity seem far from being conceptually needed in the Argument. Yet for Prior the aim of the Master Argument "was to refute the Aristotelian view that while it is now beyond the power of men or gods to affect the past, there are alternative futures between which choice is possible. Against this, Diodorus held that the possible is simply what either is or will be true" ((Prior, 1962), p.138). In other words, the Master Argument was as well an argument for determinism)[22]. As with discreteness, the very important issue here is that linearity is important to falsify a conjecture by Prior on the diodorean frame. It is for these reasons that I will assume that the diodorean logic requires linear time.

As a consequence of the above, I call **DIOD*** the logic resulting by **AG1**-**AL3**, **RG**-**RL**, the theorems of propositional classical logic and by accepting **DET**. Analogously, I call **DIOD** the modal logic obtaining by the fragment of **DIOD*** where temporal operators per se are excluded (*i.e.* the fragment where the only operators are M and L[23]). The latter is what Prior calls "the Diodorean Logic"[24].

3. FROM THE FRAME FOR S4 TO THE DIODOREAN FRAME

When one builds a logic **L**, a very natural question is: "which kind of structure does verify all and only the theorems of **L**"?. In modal logics, finding an answer to such a question means finding a *characterisation result*. After building up **DIOD**[25], he proposed a conjecture in (Prior, 1957). A wrong one, as we shall see.

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[21] It should be noticed that the linearity of time does not *imply* determinism: if we build a many-valued logic where a sentence about contingent future events is given an "undefined" truth-value, then we can endorse linearity while escaping the commitment to **DET**. Yet, as Prior points out in (Prior, 1955) (p. 211), the task is not straightforward as it seems. In addition, Diodorus and the majority of philosophers of his time seemed to adopted a two-valued logics. Even in the case of Aristotle (the main philosopher that could have been open to may values with respect to statements about the future), his endorsement of a many-value position is far from clear (for this point, see (Mariani 2009), this volume). Consequently, to the purposes of this work I will accept the idea that linearity gives a good temporal representation of determinism.

[22] In other writings, Prior confronted the Argument with non-linear (and thus indeterministic) time. He did it by reading the antecedent of **AG4** as "in every instant later than t and belonging to some branch b, p is true", probably because the Argument should have tried to reduce *this* reading to "in every instant later than t and belonging to all b, p is true". With such a reading, the Argument turns out to be false. Obviously, we know (as Prior, actually) that other readings of "at least once in the past, it is always in the future [the case that] p" make **AG4** true in non-linear time. See (Braüner & Øhrstrøm & Hasle, 2000) for this and others issue concerning Prior's reading of the Argument and non-linearity.

[23] Such a choice may look strange, since the two operators conceals temporal ones. However, in this fragment G and F may not appear alone, but just in sentences $p \wedge Gp$ or $p \vee Fp$. Since Gp and Fp cannot be disentangled by such sentences, G and F are not here acting properly as operators.

[24] Prior called such a logic **D**, but I prefer not to use that name, since it may cause confusion with the basic deontic logic, usually called **D**.

[25] A task that he accomplished in (Prior, 1955), even with some difference with my presentation.

3.1 PRIOR'S CONJECTURE

A structure characterises a (modal) logic iff the former validates all and only the theorems of the latter. To find a characterisation result may be difficult, but it cannot even be pursued without setting a precise formal machinery. In investigating the characterisation of a modal logic, Prior mainly used the device of *matrices*. Each sentence *p* is endowed with a sequence of truth-values 0 or 1. In temporal logics, we may say that this sequence represents the truth-value of *p* at the different instants in time. *Lp* (*Mp*) is given value 1 in a certain position of the sequence iff *p*'s value is 1 from that position on (at that position or some subsequent one). If a sentence is given value 1 in each position of any possible sequence of a matrix, then it is valid w.r.t. that matrix. We may say that a given matrix *characterises* a logic **L** if it validates *all and only* the theorems of **L**.

Matrices have been proven themselves in many formal results about modal logics. However, they are quite complex to handle, at least if compared with another tool that has been elaborated for the semantic of modal logic: *kripkean semantics*[26]. In these semantics, sentences are interpreted on the basis of a *Kripke frame* (or simply a "*frame*"), *i.e.* a structure made by a set of *points* and an *accessibility relation* imposed on the set. The latter determines if a given point has, so to speak, access to the information of another point.

To the sake of simplicity, here I will use *frames*, while neglecting *matrices*, since this will make the assessment of the results easier, and will achieve it by a formal tool many readers are more familiar with.

In the temporal case, sentences are interpreted on *frames* $\boldsymbol{f}$ made by sets *t* of instants and the earlier/later relation < ($\boldsymbol{f} := \langle T, <\rangle$). In order to establish the truth-value of the sentences, we use a function ν that assigns each sentence *p* a set of instants (intuitively, the set of the instants where *p* is true). We then introduce the function ρ that assigns each pair (sentence, instant) to a truth-value, according to the condition that a sentence *p* is true at the instant *t* iff $t \in \nu(p)$:

TC1 $\rho(p, t) = 1$ iff $t \in \nu(p)$
TC2 $\rho(Fp, t) = 1$ iff $\exists t' (t < t' \text{ and } \rho(p, t') = 1$
TC3 $\rho(Pp, t) = 1$ iff $\exists t' (t' < t \text{ and } \rho(p, t') = 1$

The truth-clauses for $\neg p$ or $p \circ q$ (with $\circ$ a dyadic connective) are straightforward, and the ones for *G* and *H* easily derive from **TC2** and **TC3**. A *model* based on $\boldsymbol{f}$ is a pair $\boldsymbol{m} := \langle \boldsymbol{f}, \rho\rangle$. A

[26] Such formal tools have been introduced by Saul Kripke (in (Kripke, 1959) and (Kripke, 1963)), usually considered as the founder of contemporary modal logic. Actually, before (Kripke, 1959) was published, Prior had elaborated a set of truth-clauses for tensed sentences that are similar to Kripke's semantics. This kind of semantics is also known as *possible world semantics*. Here, I prefer not to use it, since the structures employed by this semantics may be made by sets of instants, or event points of space, depending on the context where the logic has to be applied. The notion of "possible world" is then unessential to correctly refer to that semantics.

sentence p is true in (or verified by) a model ***m*** iff it is true at any instant comprised in ***m***, and false in it (falsified by it) otherwise.

Validity A sentence p is valid w.r.t. a frame f iff it is true in any ***m*** based on f.

I will also say that a frame f validates or verifies (falsifies) a sentence p if p is valid w.r.t. f (if some models based on f falsifies p). If the relation $<$ comprised in f has the property A, we will say that f is an A-frame. Since $<$ is transitive, the frames for the temporal logics are *transitive*-frames[27]. Concerning a logic **L**, I will say that

In A sentence p is *in* **L** iff p is a theorem of **L** (**L** $\vdash p$, that is, either an axiom of **L**, or the transformation of an axiom via the admitted rules of inference).

Here, it is important to notice that we need to adjust the above presentation, if we wish to deal with **DIOD** in isolation. Indeed, if we have to consider just an accessibility relation that is suitable for L, we cannot use $<$, since a frame including the earlier/later relation would not verify **AL2**. Instead, we have to use $\leq$, the "earlier/later (or identical)" relation. We may think of $\leq$ as imposed on the set t of instants I have mentioned above. Thus we have that $f_{\mathbf{DIOD}}$ is $\langle T, \leq \rangle$, and the truth-clause for Lp is :

TCL $\rho(Lp, t) = 1$ iff $\forall t'$ ($t \leq t'$ then $\rho(p, t')) = 1$

the clause can be easily shown to be equivalent to the one for $p \wedge Gp$ if the relation of the frame is $<$. The truth-clause for M is straightforward (since M is $\neg L \neg$). The problem we will address on this section is: which frame is a frame *for* **DIOD**? This meaning nothing but "which frame *characterises* **DIOD**?" Some technical notions are helpful here:

For 1 A frame f is the frame for a logic **L** ($f_{\mathbf{L}}$) iff f characterises **L** (relatively to a given language l).

For 2 The frame for a logic **L** is the frame for a logic **L'** iff it is the frame for **L** and it is the frame **L'**.[28]

It is clear that the frames for **DIOD** are reflexive and transitive (since $\leq$ is). In (Prior, 1955) (p. 209), Prior had already -correctly- guessed that the diodorean frame verifies *all* the

[27] A remarkable exeption is the frame for the *minimal* temporal logic, whose theorems do not include a sentence expressing transitivity. However, the temporal reading of such a logic is somehow questionable.

[28] Please notice that the last definition does not imply that **L** and **L'** coincide: indeed, they may be based on two different languages, and thus the former has f as its frame relatively to the language **L** while the latter has f as its frame relatively to the language **L'**. However, if **L** and **L'** are based on the same languages and f is the frame for both, then **L** and **L'** coincide (since they validate the same sentences).

theorems of **S4**, since the relation in f_{S4} is reflexive and transitive. These conditions correspond in the logic to **AL2** and **AL3**, that is to $Lp \rightarrow p$ and $Lp \rightarrow LLp$ respectively. In addition, **AL1** $((L(p \rightarrow q) \wedge Lp) \rightarrow Lq)$ is valid w.r.t. to f_{S4}. Since the rules of inference are shared by the two logics and preserve validity, all theorems of **DIOD** are verified by f_{S4}.

In (Prior, 1957) Prior tries to go look beyond this simple result. There, he conjectures that that frame verified *all and only* the theorems in **DIOD**. Rephrasing Prior's investigation in the terminology and by the tools employed in this paper, we have the following conjecture:

Prior's Conjecture: The frame for **S4** is the frame for the Diodorean modalities: $f_{DIOD} = f_{S4}$.

The original point of Prior's Conjecture is stating that *only* the theorems of **DIOD** are verified by f_{S4}.

With our current knowledge of the frames for modal logic, it is not difficult to foresee that the conjecture is incorrect. However, it was a reasonable option at those times. Indeed, when Prior was studying the diodorean modalities, the only known logic between **S4** and **S5** was **S4.5**. Prior knew that such a logic includes a sentence that has no plausible diodorean reading[29]. Thus, the frame for **S4.5** had been immediately excluded. In addition, in those very years **S4.5** was later found equivalent to **S5** (thus there exists no "frame for" **S4.5**). The frame for **S5** does not go, since the latter includes $Mp \rightarrow LMp$, and such a sentence is clearly false in a diodorean reading[30]. The only candidate left was **S4**[31].

3.2 COUNTEREXAMPLES: FROM THE FRAMES FOR S4 TO THE FRAMES FOR DIOD

As we have seen, **DIOD** was designed by Prior to be a deterministic logic, on the basis of the idea that **DET** was essential in the diodorean conception of modalities. It turns out that the principle, though very vague, has been enough to expose Prior's Conjecture to relevant counterexamples. Let us consider the following sentence:

lin $Mp \wedge Mq \rightarrow (p \wedge q) \vee M(p \wedge Mq) \vee M(q \wedge Mp)$

It is easy to prove that ***lin*** is valid in a frame where the accessibility relation is transitive *and linear*. Take a linearly ordered set of instants: if $Mp \wedge Mq$ is true at t, then either $p \wedge q$ is true at

[29] For this, see (Prior, 1967), p.23-24.

[30] The fact that now or in the future p is true, does not imply that the same holds for every future instant. If P is true now and false thereafter, Mp is true, while MLp is false.

[31] It should also be considered that modal logic and its formal results were then at their beginnings, and many issues, though looking obvious today, were still hypothesis waiting for a proof or a counterexample. In addition, the device of matrices makes it harder to find counterexamples as the one we have presented. While it is easy for a single researcher to find all them using *frames* and *models*, a much more articulated work is needed if using *matrices*, and just the contribution of many researchers may help to find counterexamples in a short time.

t itself, or the instant that verifies *p* (or *q*) is earlier than the one that verifies *q* (or *p*), or identical to it. This possible combinations give us the consequent of ***lin***.

To see that a non linear frame falsify ***lin***, suppose there is an instant *t* where $M(p \wedge q)$ is true and $(p \wedge q)$ is not. Now take two later instants *t′* and *t′′* that are incomparable (they are not earlier, later or identical one with another), satisfying the following:

(i) In all the instants between *t* and *t′* (both excluded), $\neg p \wedge \neg q$ is true. The same at all the instants between *t* and *t′′* (both excluded).
(ii) At *t′* we have that *p* is true but *q* is false thenceforth (thus having that $L\neg q$ is true at *t′*).
(iii) At *t′′* we have that *q* is true but *p* is false thenceforth (thus having that $L\neg p$ is true at *t′′*).

Since *t′* and *t′′* are incomparable, ***(ii)***-(***iii)*** are compatible one with another. But as a consequence of ***(i)***-***(iii)***, our sentence is false. Indeed, $Mp \wedge Mq$ is true at *t* (because *p* and *q* are true at *t′* and *t′′*, respectively), but $(p \wedge q) \vee M(p \wedge Mq) \vee M(q \wedge Mp)$ is false at *t* (since $p \wedge q$ is and no instant from *t* on verifies $(p \wedge Mq)$ or $(q \wedge Mp)$). The counterexample shows as well that there are transitive but not linear. This has tow main consequences.

(l.1) ***lin*** is not in **S4**. Otherwise, the implication from **AG2** to ***lin*** should be in **S4**. But this does not hold, since some transitive frame falsities ***lin***.
(l.2) ***lin*** is not valid w.r.t. f_{S4}, since there is a model that is *transitive* and yet falsifies ***lin*** (and since f_{S4} validate all and only the sentences in **S4**).

Prior's attention on ***lin*** was first driven by (Hintikka, 1958) (a review of (Prior, 1957)), where it is suggested that a temporal interpretation of **S4** cannot by given without adding ***lin*** to it[32]. In any case, ***(l.1)*** leads to the conclusion that $f_{DIOD} \neq f_{S4}$: the frame for **S4** is not the diodorean frame. This

(l.3) led Prior to dismiss his own conjecture in (Prior, 1958), where he explicitly admit that ***lin*** must be in **DIOD** (in accordance with the links between linearity and **DET**, see section 2.1)[33].

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[32] ***lin*** is not the only sentence that readdresses the search for the diodorean frame toward linear frames: $L(Lp \rightarrow Lq) \vee L(Lq \rightarrow Lp)$ (***lin****) requires linearity as well to be valid. The sentence had been pointed out to Prior by Lemmon (see (Prior, 1958), p.226). Prior later proved that ***lin*** and ***lin**** are equivalent ((Prior, 1964)) and that ***lin**** is valid in **DIOD**. The last proof seems to assume that linearity as a condition that is plausible for time *in se*, even out of the diodorean conception of modality.

[33] Actually, Prior's position about ***lin*** is somehow unclear: in (Prior, 1958) and (Prior, 1967), he defends its endorsement in **DIOD** because of its intrinsic "tense-logical plausibility". A consequence is that a linear (and hence deterministic) representation of time is imposed not by the diodorean logic, but by what our intuitions about time take to be plausible. If one argues this way, linearity should be suitable for any temporal logic (**DIOD** included). However, in this way the Master Argument and the diodorean

(l.4) helped to understand that there is a logic that is stronger than **S4** and yet weaker than **S5**. Indeed, ***lin*** cannot be derived by any axiom of **S4** (see ***(l.2)*** above). At the same time, no axiom of **S5** can be derived by it.

A new modal logic was *de facto* discovered through the falsification of Prior's Conjecture. The new logic was called **S4.3** (today the most widespread name for it). Establishing the fatherhood of the logic is beyond the purpose of this paper. In any case, it should be case that at least two works reached to ***lin*** (or equivalent sentences). One is Hintikka, that simply mention it as a sentence that is not in **S4** (see above), the other is actually a duo: Michael Dummett and Edward Lemmon, that in (Dummett & Lemmon, 1959) found the sentence independently from Hintikka and gave the name to **S4.3**. The interesting thing to notice is that the work by Dummett and Lemmon focus on *intermediate modal logics*, and that its rationale is completely independent from Prior's research. Indeed, the two authors focused on intermediate modal logics because they can be used for establishing properties of logics that are stronger than the Intuitionistic one but weaker than the Classical one[34]. Finding out that ***lin*** is not in **S4** has been useful for finding one of such logics and extending the class of modal logics.

Thus, the same discovery had led to a progress both in the search of the diodorean frame and in our knowledge of intermediate modal logics. The philosophical topic of the diodorean logic has benefited from research that was undertook for more specific and technical reasons.

(Dummett & Lemmon, 1959) crosses with the search of the diodorean frame also in another way: it is the first study where it is noticed that $f_{S4.3}$ is not *discrete*. This is important for us, since the diodorean logic should go together with the second condition Prior added to the Argument (that is *discreteness*).
Now let us take the sentence:

disc $(MLp \wedge (L(\neg p \rightarrow M(p \wedge M\neg p)) \rightarrow p$

It is easy to see that if $\leq$ is non-discrete, the sentence is false, while the discreteness of $\leq$ makes it true[35]. Indeed, take the situation:

(i') There is an instant t that verifies both MLp and $\neg p$.
(ii') There is an instant t' such that $t \leq t'$ and that verifies Lp.
(iii') At any instant from t on, $\neg p \rightarrow M(p \wedge M\neg p)$.

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conception of modality would cease to be a relevant argument and conception for determinism, in contrast with (Prior, 1962), p.138.

[34] This field of study has its roots in the Gödel-Tarski-McKinsey theorem, that states that a sentence p is a theorem of Intuitionistic Logic iff its modal translation is a theorem of **S4**. In those years one of the main works on the topic was (Dummett, 1959).

[35] In (Dummett & Lemmon, 1959) the relevant sentence is: (***disc****) $(L(L(p \rightarrow Lp) \rightarrow p) \wedge MLp) \rightarrow p$. The equivalence of ***disc*** and ***disc**** has been proven by Prior in (Prior, 1967).

Now, $M(p \wedge M\neg p)$ is true at t, by ***(i')*** and ***(iii')***. As a consequence, there must be an instant t'' that verifies $p \wedge M\neg p$. Such an instant is later than t, since the latter falsifies p. But it is also earlier than t', since $M\neg p$ is always false from t' on. For the same reason, there is an instant t''' *between* t' and t'' where $\neg p$ is true: it cannot be t'', t' or any instant later than t'', since they all verify P. At the same time, it must be later than t', in order $p \wedge M\neg p$ to be true there. But in t''', $M(p \wedge M\neg p)$ is true, by ***(i')*** and the fact that t''' verifies $\neg p$. As a consequence, a further instant (strictly) *between* t''' and t' is needed, and so *ad infinitum*. This is perfectly consistence with density and continuity, since between any two instant there are infinite instant. Hence the situation may hold in frames that are based on a dense or continuous $\leq$. Thus proves that ***disc*** is not valid w.r.t. non-discrete frames. On the contrary, if $\leq$ is discrete there will be a last instant between t''' and t'. In this last instant, even if having $\neg p$, $M(p \wedge M\neg p)$ cannot be but false, since the instant is followed by t', where Lp is true. As a consequence, if we have MLp and $L(\neg p \rightarrow M(p \wedge M\neg p)$ at t, we must also have p at t. This shows the validity of ***disc*** w.r.t. discrete frames. This means that:

(d.1) ***disc*** is not in **S4.3** (for reasons analogous to the ones in ***l.1***).

(d.2) ***disc*** is not valid w.r.t. $f_{S4.3}$, since there is a model that is *transitive*, *linear* and yet falsifies ***lin*** (and since $f_{S4.3}$ validate all and only the sentences in **S4.3**).

As a consequence, $f_{S4.3}$ is not the diodorean frame. Here, we have a situation that resembles the one we had with ***lin***: a new logic was discovered. Or better, it had been clarified what axioms **DIOD** needs. And once again, the investigation on diodorean modalities had benefited from some other researches, namely those on intermediate logics.

However, at this point discreteness is the only condition to be unfulfilled. Thus, it is enough to added discreteness to a reflexive, transitive and linear frame to have f_{DIOD}. This is what Prior implicitly suggests in (Prior, 1967), p.29. It is clear that such a new frame validate *all* the theorems of **DIOD**. But does it validate *only* them? In other words: is it a frame *for* **DIOD**. Prior does not prove it in (Prior, 1967), but anyway that was not a conjecture at that time. Robert Bull had already proved in (Bull, 1965) that the frame for **DIOD** is discrete, reflexive, transitive and linear[36]. As a consequence, we may say that (Prior, 1967) (p.29) concludes the search for the diodorean frame.

Few time later, **DIOD** resurfaced in the research on intermediate logics. In (Zeman, 1968) the logic is introduced (together with a cognate logic) with the name most often used today: **S4.3.1**. It was already clear that $f_{S4.3.1}$ was discrete. In any case, the success of the name is well deserved, since it helps in immediately grasping the place **DIOD** has in the logics between **S4** and **S5**.

[36] Two further different proofs of that are given in an unpublished work by Kripke and in (Segerberg, 1970).

4. THE SEARCH FOR THE DIODOREAN FRAME AND MODAL LOGIC

We have seen that the search for the diodorean frame has benefited from two different researches in modal logics: (i) the research on intermediate logics, and (ii) the research for characterisation results. Thus shows how the work in progress on technical issues of logics helped Prior's investigation.

It is now time to see how Prior investigation stimulated some technical result. We may distinguish two different contributions: (1) indirect ones (mainly to characterisation results), (2) stimulus to works that explicitly mention diodorean modalities. Let look at them separately.

(1) The direction of the benefits has not been just *from* characterisation results *to* the search of diodorean frame. Prior's conjecture has promoted some researches *in* that field.

Kripke, in private correspondence, presented to Prior a matrix *for* **S4**, and that resembles some frames for branching time. The issue is mentioned in (Prior, 1967), p.27, and discussed in detail in (Øhrstrøm & Hasle, 1993). Kripke was also able to find a characterisation result for **S4.3.1**, and contributed as well to the falsification of Prior's Conjecture with finding that $LMp \vee LM\neg p$ that is not valid in the frames for **S4** (the proof is straightforward and so I omit it).

As clear from the same correspondence, Kripke's interest in the characterisation for this kind of logics is rooted in his reading of (Prior, 1957), and on the philosophical relevance of a temporal interpretation of some modal logics. In particular, Kripke thought that temporal specifications are not relevant in scientific theories[37]. This shows that his interest to such logics was linked to the philosophical issues Prior has addressed by using formal methods about modalities and temporal specifications.

Another result came from Lemmon. In (Dummett & Lemmon, 1959) he presented a modification of Kripke matrix to verify all and only the theorems of **S4.2**, that is **S4** plus $MLp \rightarrow LMp$. In (Prior, 1967), Prior presents the sentence as a result of Lemmon's own work, and as preceding the work with Dummett. We may hypothesise a connection between Lemmon's matrix and Prior's work. Lemmon interest in modal logic was triggered by (Prior, 1957)[38], and in addition $MLp \rightarrow LMp$ had a role in the search of the diodorean frame, since it is valid in all linear frames and is falsified by the frame for **S4**[39].

(2) In addition, some works in modal logic take **DIOD** explicitly into account. Examples of this are (Bull, 1965) (already mentioned) and (Makinson, 1966). Beside proving Bull's paper undertakes an algebraic treatment of all the logics that had been involved in Prior's search (**S4**,

[37] see (Øhrstrøm & Hasle, 1993).
[38] To be more precise, it was triggered by the John Locke Lectures that Prior delivered in Oxford (1956).
[39] Indeed, $MLp \rightarrow LMp$ expresses the condition of *convergence*, that is implied by linearity (while the converse does not hold) and it is not implied by transitivity.

S4.3, and obviously **DIOD**). (Makinson, 1966) shows that infinite non-equivalent formulae are contained in the sentences in **S4.2**, **S4.3** and **DIOD**[40], as it is for **S4**. Many years later, Robert Goldblatt applied the notion of diodorean modalities to Minkowski spacetime (see (Goldblatt, 1980)), finding some interesting characterisation results, with the collaboration of Johan Van Benthem.

We may now sum up what has emerged through the paper. The search of the diodorean frame has entwined with research of other fields of modal logics through:

(1) benefits from the research on intermediate logics, as witnessed by the fact that works in that field contributed to falsify Prior's Conjecture ((Dummett & Lemmon, 1959)).

(2) interaction with characterisation results, as witnessed by the fact that (a) the result in (Bull, 1965) ensures that the frame for **DIOD** is reflexive, transitive, linear and discrete, a result that Prior acknowledged in (Prior, 1967), p.31, (b) the research on characterisation results for **S4** and **S4.2** by Kripke and Lemmon (respectively) was probably motivated by Prior's Conjecture or by other issues addressed by Prior.

(3) explicit consideration in technical works on modal logics, as shown by a variety of studies that focuses on logics that extend **S4**. In these studies, the modalities under account are called "diodorean modalities".

5. CONCLUSION

In this paper, I argued that the search for the diodorean frame entwined with the researches on intermediate logics and on characterisation results, that it has benefited from this, and that in some cases stimulated them. Thus, the history of the diodorean modalities can be taken as a fruitful case of interaction between philosophy and logic, and as an example of how philosophical topics have interacted with technical investigations in modal logic.

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[40] Remarkably, this opposes these logics to their non-modal counterparts between Intuitionistic and Classic logic, as Makinson himself stresses ((Makinson, 1966), p. 406).

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